ADDING A RANDOM REAL NUMBER AND ITS EFFECT ON MARTIN'S AXIOM

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Abstract. We show that adding a random real number destroys a large fragment of Martin's axiom, namely Martin's axiom for partial orders that have precalibre-$\aleph_1$, thus answering an old question of J. Roitman [9]. We also answer a question of J. Steprans and S. Watson [13] by showing that, by a forcing that preserves cardinals, one can destroy the precalibre-$\aleph_1$ property of a partial ordering while preserving its ccc-ness.

J. Roitman [9] showed that adding a random real to a model of set theory produces two ccc topological spaces such that their product is not ccc, hence Martin's Axiom for $\aleph_1$ (henceforth denoted by $MA_{\aleph_1}$) does not hold in the generic extension. Indeed, many consequences of $MA_{\aleph_1}$ are lost after adding a random real. For example, as shown by K. Kunen (see [9]), adding a random real yields a topological space all whose finite products are L-spaces, a space that cannot exist under $MA_{\aleph_1}$. Further examples are given by S. Todorčević. For instance, he shows that adding a random real adds an entangled set of reals, which cannot exist under $MA_{\aleph_1}$ (see [14], 6.10).

In spite of this, a good fragment of $MA_{\aleph_1}$ is preserved by adding a random real, namely $MA_{\aleph_1}(\sigma\text{-linked})$ (a result of Kunen first published in [9], and later disclaimed in [10], but see [2] for a proof). Other important consequences of $MA_{\aleph_1}$ that do not follow from $MA_{\aleph_1}(\sigma\text{-linked})$ are also preserved by adding a random real. For example, the fact that every Aronszajn tree on ω_1 is special, hence the Suslin's Hypothesis (R. Laver [8]), and the fact that every (ω_1, ω_1)-gap on $\mathcal{P}(\omega)/Fin$ is indestructible (J. Hirshorn [5]). So a natural question is how much of $MA_{\aleph_1}$ is preserved by adding a random real. In particular, Roitman [9] asks if adding a random real preserves $MA_{\aleph_1}(\text{Precalibre-}\aleph_1)$, or even $MA_{\aleph_1}(\text{Property-}\mathcal{K})$. A related question, asked in [1], is if $MA(\sigma\text{-centered})$ plus "Every Aronszajn tree is special" implies $MA(\text{Productive-ccc})$. We answer both questions in the negative by exhibiting a Random-name $\underset{\sim}{\mathbb{P}}$ for a precalibre-$\aleph_1$ partial ordering together with a Random-name for a family of $\aleph_1$-many dense open subsets of $\underset{\sim}{\mathbb{P}}$ such that, assuming the ground model satisfies $MA_{\aleph_1}(\sigma\text{-linked})$, in the Random-generic extension there is no filter on $\underset{\sim}{\mathbb{P}}$ generic for the family.

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In the second part of the paper we answer a question of J. Steprans and S. Watson [13] by showing that it is consistent, modulo ZFC, that the CH holds and there exist a forcing notion T of cardinality $\aleph_1$ that preserves ω_1, and two precalibre-$\aleph_1$ partial orderings, such that forcing with T preserves their ccc-ness, but it also forces that their product is not ccc and therefore they don't have precalibre-$\aleph_1$.

1. Preliminaries

Recall that a partially ordered set (or poset) $\mathbb{P}$ is *ccc* if every antichain of $\mathbb{P}$ is countable; it is *powerfully-ccc* if every finite power of $\mathbb{P}$ is ccc; it is *productive-ccc* if the product of $\mathbb{P}$ with any ccc poset is also ccc; has Property-$\mathcal{K}$ (or is *Knaster*) if every uncountable subset of $\mathbb{P}$ contains an uncountable subset consisting of pairwise compatible elements; it has *precalibre*-$\aleph_1$ if every uncountable subset of $\mathbb{P}$ has an uncountable subset consisting of finite-wise compatible elements; it has *calibre*-$\aleph_1$ if every uncountable subset of $\mathbb{P}$ contains an uncountable subset for which there exists a condition stronger than all its elements; it is σ*-linked* if it can be partitioned into countably-many pieces so that each piece is pairwise compatible; and it is σ*-centered* if it can be partitioned into countably-many pieces so that each piece is finite-wise compatible. We have the following implications:

$$\sigma\text{-centered} \Rightarrow \sigma\text{-linked} \Rightarrow \text{Knaster} \Rightarrow \text{Productive-ccc} \Rightarrow \text{Powerfully-ccc} \Rightarrow \text{ccc},$$

$$\sigma\text{-centered} \Rightarrow \text{Precalibre-}\aleph_1 \Rightarrow \text{Knaster},$$

$$\text{Calibre-}\aleph_1 \Rightarrow \text{Precalibre-}\aleph_1.$$

and these are the only implications that can be proved in ZFC.

For a class of ccc posets satisfying some property Γ, *Martin's Axiom for* Γ, denoted by $MA_{\aleph_1}(\Gamma)$, asserts: for every $\mathbb{P} \in \Gamma$ and every family $\{D_\alpha : \alpha < \omega_1\}$ of dense open subsets of $\mathbb{P}$, there exists a filter $G \subseteq \mathbb{P}$ that is *generic* for the family, that is, $G \cap D_\alpha \neq \emptyset$ for every $\alpha < \omega_1$.

Thus, $MA_{\aleph_1}(ccc) = MA_{\aleph_1}$, and we have the following implications:

$$MA_{\aleph_1} \Rightarrow MA_{\aleph_1}(\text{Powerfully-ccc}) \Rightarrow MA_{\aleph_1}(\text{Productive-ccc}) \Rightarrow$$

$$\Rightarrow MA_{\aleph_1}(\text{Knaster}) \Rightarrow MA_{\aleph_1}(\sigma\text{-linked}) \Rightarrow MA_{\aleph_1}(\sigma\text{-centered}),$$

$$MA_{\aleph_1}(\text{Knaster}) \Rightarrow MA_{\aleph_1}(\text{Precalibre-}\aleph_1) \Rightarrow MA_{\aleph_1}(\sigma\text{-centered}),$$

$$MA_{\aleph_1}(\text{Precalibre-}\aleph_1) \Rightarrow MA_{\aleph_1}(\text{Calibre-}\aleph_1).$$

For all the facts mentioned in the rest of the paper without a proof, as well as for all undefined notions and notations, see [6].

1.1. **Random forcing.** Recall that the *Random* partial ordering consists of all Borel subsets of Cantor space 2^ω of positive Lebesgue measure, ordered by $\subseteq$. *Random* is σ-linked. If a filter $G \subseteq$*Random* is generic over some transitive model M, then $\bigcap G$ consists of a single real number, called a *random real over* M. The following lemma is due to Solovay (see [15], Lemma 5.1, for a more general result).

Lemma 1. *If $MA_{\aleph_1}(\sigma$-linked$)$ holds, then the Random poset has calibre-$\aleph_1$. That is, for every uncountable subset X of Random, there exists a condition p such that the set $\{q \in X : p \leq q\}$ is uncountable.*

Proof. For each $n > 1$, let $\mathbb{P}_n$ be the subset of Random consisting of all the closed sets of measure greater than $\frac{1}{n}$. Let $\mathbb{P} = \prod_{n>1} \mathbb{P}_n$, with finite support. $\mathbb{P}$ is σ-linked. Fix a subset X of Random of cardinality $\aleph_1$. For each $p \in X$, the set

$$D_p := \{q \in \mathbb{P} : \exists n(q(n) \leq p)\}$$

is dense open in $\mathbb{P}$. By $MA_{\aleph_1}(\sigma\text{-linked})$ there exists a filter $G \subseteq \mathbb{P}$ that is generic for the family $\{D_p : p \in X\}$. For some n, $X \cap G(n)$ is uncountable. Letting $p_n = \bigcap G(n)$, we have that p_n is a closed set that has measure at least $\frac{1}{n}$, hence it belongs to Random, and the set $\{p \in X : p_n \leq p\}$ is uncountable. $\square$

Recall that, for every $0 < \epsilon < 1$, the $Amoeba_\epsilon$ partial ordering consists of all Borel subsets of Cantor space of Lebesgue measure $> \epsilon$, ordered by $\subseteq$. $Amoeba_\epsilon$ is also σ-linked. If a filter $G \subseteq Amoeba_\epsilon$ is generic over some transitive model M, then $\bigcap G$ consists of a set of reals of measure ϵ, all whose elements are random reals over M.

2. A name for a partial ordering

Suppose $\mathbb{Q}$ is a forcing notion that preserves ω_1, and $\underset{\sim}{r}$ is a $\mathbb{Q}$-name for a real, that is,

$$\Vdash_{\mathbb{Q}} \text{“}\underset{\sim}{r} : \omega \to 2\text{”}.$$

We will define a $\mathbb{Q}$-name $\underset{\sim}{\mathbb{P}} = \underset{\sim}{\mathbb{P}}_{\mathbb{Q},\underset{\sim}{r}}$ for a forcing notion.

Definition 2. *Let $\Omega := \{\delta : \delta < \omega_1,\ \delta \text{ a limit }\}$. For $\delta \in \Omega$, let $e_\delta \subseteq \delta = sup(e_\delta)$, with e_δ of order-type ω. For $\delta \in \omega_1 \setminus \Omega$, let $e_\delta = \emptyset$.*

Define $\pi : [\omega_1]^2 \to \omega$ by $\pi\{\alpha, \beta\} = |\alpha \cap e_\beta|$, whenever $\alpha < \beta < \omega_1$.

Let $\eta_{\alpha,n} \in 2^\omega$, for $\alpha < \omega_1$ and $n < \omega$, be pairwise distinct.

Let $\langle k_\sigma : \sigma \in 2^{<\omega}\rangle$ be a list of natural numbers with no repetitions.

For $\rho \in 2^\omega$, let $\underset{\sim}{s}_\rho$ be the following $\mathbb{Q}$-name for a real:

$$\underset{\sim}{s}_\rho = \langle \underset{\sim}{r}(k_{\rho\restriction l}) : l < \omega\rangle.$$

(For simplicity of notation, in the sequel we shall write $\underset{\sim}{s}_{\alpha,n}$ for $\underset{\sim}{s}_{\eta_{\alpha,n}}$.)

Let $\underset{\sim}{\mathbb{P}}$ be the set of all u such that:

1. *u is a function with domain a finite subset of ω_1 and range included in ω.*
2. *If $u(\alpha_1) = u(\alpha_2) = n$, $\alpha_1 < \alpha_2$, and $m = \pi\{\alpha_1, \alpha_2\}$, then*

$$\underset{\sim}{s}_{\alpha_1,n} \restriction m = \underset{\sim}{s}_{\alpha_2,n} \restriction m.$$

The ordering $<_{\mathbb{P}}$ on $\mathbb{P}$ is the reversed inclusion $\supset$.

Lemma 3. *If $\mathbb{Q}$ has calibre-$\aleph_1$, then $\underset{\sim}{\mathbb{P}}$ has precalibre-$\aleph_1$ in $V^{\mathbb{Q}}$.*

Proof: Suppose $p_* \Vdash_{\mathbb{Q}}$ “$\langle \underset{\sim}{u}_\xi : \xi < \omega_1\rangle$ is a sequence of elements of $\underset{\sim}{\mathbb{P}}$”. We shall find $p_1 \leq p_*$ and $X \in [\omega_1]^{\aleph_1}$ such that

$$p_1 \Vdash_{\mathbb{Q}} \text{“}\bigcup\{\underset{\sim}{u}_\xi : \xi \in x\} \in \underset{\sim}{\mathbb{P}}, \text{ for every finite } x \subseteq X\text{”}.$$

This suffices.

Since $\mathbb{Q}$ has calibre-$\aleph_1$, we may assume that p_* decides $\underset{\sim}{u}_\xi$, for all ξ. So, we may also assume that

$$\underset{\sim}{u}_\xi = u_\xi = \{(\alpha_{\xi,l}, n_{\xi,l}) : l < l^*\}$$

for some fixed l^*, and the $\alpha_{\xi,l}$ are strictly increasing with respect to l. Moreover, we may assume that for each $l < l^*$, $n_{\xi,l}$ is equal to some fixed n_l, for all $\xi < \omega_1$.

We may now assume that the u_ξ form a Δ-system with root u^*, and therefore we may further assume that if $\zeta < \xi < \omega_1$, $\alpha \in dom(u_\zeta \setminus u^*)$ and $\alpha' \in dom(u_\xi \setminus u^*)$, then $\alpha < \alpha'$. Finally, we may assume that r^* is an initial segment of u_ξ, for every ξ. That is,

$$u_\xi = u^* \cup \{(\alpha_{\xi,l}, n_l) : l \in [l_1, l^*)\}$$

for some fixed $l_1 < l^*$.

For each $\xi < \omega_1$, let $h(\xi)$ be the least ξ' such that for every $\zeta < \xi'$, $\alpha_{\zeta,l} < \xi'$, all $l \in [l_1, l^*)$. Note that h is strictly increasing and continuous. Hence, h has a club C of fixed points.

For each $\xi < \omega_1$, let $F(\xi)$ be the least element of C greater than ξ. Define

$$u'_\xi := u^* \cup \{(\alpha_{F(\xi),l}, n_l) : l \in [l_1, l^*)\}.$$

Let C' be the set of limit points of C. We claim that the following holds for every $\xi \in C'$:

(1) If $\zeta < \xi$, then $\alpha_{F(\zeta),l} < \xi$, all $l \in [l_1, l^*)$.
(2) $u'_\xi \cap ((\xi + 1) \times \omega) = u^*$.

For (1), notice that since $\xi \in C'$, $F(\zeta) < \xi$, hence $F(\zeta) < h(\xi)$, which implies that $\alpha_{F(\zeta),l} < \xi$. For (2), notice that the $\alpha_{\zeta,l}$, for $\zeta < \xi$ are cofinal in ξ. Thus, $\alpha_{\xi,0} \geq \xi$. Hence, since $F(\xi) > \xi$, we have that $\alpha_{F(\xi),0} > \xi$.

Let $f : C' \to \omega$ be defined by:

$$f(\xi) = sup\{k : \pi\{\alpha, \alpha'\} = k,\ \alpha \in dom(u'_\zeta),\ \alpha' \in dom(u'_\xi),\ \zeta < \xi\}$$

Note that f is well-defined because $dom(u'_\xi \setminus u^*) \cap \xi + 1 = \emptyset$ (by (2)) and $dom(u'_\zeta) \subseteq \xi$, for all $\zeta < \xi$ (by (1)), and so the set

$$\{e_\alpha \cap \alpha_{F(\zeta),l} : \alpha \in dom(u'_\xi \setminus u^*), \zeta < \xi, l \in [l_1, l^*)\}$$

is finite. Thus, there exists $k^* \in \omega$ such that for some stationary $S \subseteq C'$, for all $\xi \in S$, $f(\xi) = k^*$. By the Pigeonhole principle, and since $\mathbb{Q}$ has calibre-$\aleph_1$, there exist some $p_1 \geq p_*$ and a stationary $S^* \subseteq S$ such that

$$p_1 \Vdash \text{“}\langle \underset{\sim}{s}_{\alpha_{F(\xi),l}, n_l} \restriction k^* : l \in [l_1, l^*)\rangle \text{ is the same for all } \xi \in S^*\text{”}.$$

Letting $X := F[S^*]$, we have that

$$p_1 \Vdash_{\mathbb{Q}} \text{“}\bigcup\{\underset{\sim}{u}_\xi : \xi \in x\} \in \underset{\sim}{\mathbb{P}}, \text{ for every finite } x \subseteq X\text{”}$$

as required. $\square$

Fact 4. *In $V^{\mathbb{Q}}$, for each $\delta \in \Omega$, the set $\underset{\sim}{I}_\delta = \{u \in \underset{\sim}{\mathbb{P}} : \delta \in dom(u)\}$ is a dense open subset of $\underset{\sim}{\mathbb{P}}$.*

Proof. Suppose $u \in \underset{\sim}{\mathbb{P}}$, $u \notin \underset{\sim}{I}_\delta$. Let $m \in \omega$ be such that $m \notin range(u)$ and let $u' := u \cup \{(\delta, m)\}$. Then $u' \in \underset{\sim}{\mathbb{P}}$ and $u' < u$. $\square$

3. Adding a random real

Let now $\mathbb{Q}$ be the *Random* forcing, and let $\underset{\sim}{r}$ be the canonical $\mathbb{Q}$-name for the generic random real. Let $\underset{\sim}{\mathbb{P}} = \mathbb{P}_{\mathbb{Q},\underset{\sim}{r}}$.

Lemma 5. *Suppose $MA_{\aleph_1}$(σ-linked) holds. Then in $V^{\mathbb{Q}}$ there is no directed $\underset{\sim}{G} \subseteq \underset{\sim}{\mathbb{P}}$ such that for all $\delta \in \Omega$, $\underset{\sim}{I}_\delta \cap \underset{\sim}{G} \neq \emptyset$. In fact, there are no $\underset{\sim}{m}$ and stationary $\underset{\sim}{S} \subseteq \omega_1$ such that $\underset{\sim}{S} \times \{\underset{\sim}{m}\} \subseteq \bigcup \underset{\sim}{G}$.*

Proof. Clearly, the first statement follows from the second. So, we prove the second statement. We work in V. Suppose $p_0 \in \mathbb{Q}$ is such that

$$p_0 \Vdash_{\mathbb{Q}} \text{“}\underset{\sim}{m}, \underset{\sim}{S}, \underset{\sim}{G} \text{ are a counterexample”}.$$

Without loss of generality, $\underset{\sim}{m} = m$, i.e., p_0 decides $\underset{\sim}{m}$.

Claim 6. *We may assume $\underset{\sim}{S} = S$, where S is some stationary subset of ω_1 in V.*

Proof of the claim: Let $S := \{\delta \in \Omega : p_0 \nVdash \delta \notin \underset{\sim}{S}\}$. Clearly, S is stationary. For $\delta \in S$, let $p_{1,\delta} \in \mathbb{Q}$, $p_{1,\delta} \leq p_0$, be such that $p_{1,\delta} \Vdash \delta \in \underset{\sim}{S}$. We can find a stationary $S' \subseteq S$ and $0 < n < \omega$ such that for every $\delta \in S'$, $\mu(p_{1,\delta}) > \frac{1}{n}$. *Amoeba*$_{\frac{1}{n}}$ is σ-linked, and so is the ω-product of *Amoeba*$_{\frac{1}{n}}$. Hence, using MA_{ω_1}(σ-linked), for each $k \in \omega$, we can find $p_{2,k}$ and S_k such that for every $\delta \in S_k$, $p_{2,k} \leq p_{1,\delta}$, and $S = \bigcup_{k<\omega} S_k$. So, for some k, S_k is stationary. This proves the claim. $\square$

Let

$$T := \{\sigma \in 2^{<\omega} : \text{for stationary-many } \delta \in S,\ \sigma \subseteq \eta_{\delta,m}\}.$$

Notice that T is a perfect subtree of $2^{<\omega}$, and that $T \in V$.

For each $\sigma \in T$, let S_σ denote the stationary set of $\delta \in S$ for which $\sigma \subseteq \eta_{\delta,m}$. Let S'_σ be the club set of countable limit points of S_σ, and let $E := \bigcap\{S'_\sigma : \sigma \in T\}$. Thus, E is a club subset of ω_1.

Pick $\delta^* \in S \cap E$, and let $e_{\delta^*} = \{\alpha_n : n < \omega\}$.

Let ρ be a branch of T different from $\eta_{\delta^*,m}$. So, for every n, there is $\delta_n \in S_{\rho\restriction n} \cap \delta^* - \alpha_n$. Thus, $\pi\{\delta_n, \delta^*\} \geq n$.

Since δ_n, $\delta^* \in S$, p_0 forces that (δ_n, m) and (δ^*, m) are in $\bigcup \underset{\sim}{G}$. Hence, since $\underset{\sim}{G}$ is directed, by the definition of $\underset{\sim}{\mathbb{P}}$ we must have:

$$\underset{\sim}{s}_{\delta_n,m} \restriction n = \underset{\sim}{s}_{\delta^*,m} \restriction n$$

because $n \leq \pi\{\delta_n, \delta^*\}$.

And since $\rho \restriction n = \eta_{\delta_n,m} \restriction n$, we have:

$$\underset{\sim}{s}_\rho \restriction n = \underset{\sim}{s}_{\delta_n,m} \restriction n.$$

Hence,

$$\underset{\sim}{s}_\rho \restriction n = \underset{\sim}{s}_{\delta^*,m} \restriction n$$

for every n. Thus,

$$\underset{\sim}{s}_\rho = \underset{\sim}{s}_{\delta^*,m}.$$

But $\rho \neq \eta_{\delta^*,m}$, and $\langle k_\sigma : \sigma \in 2^{<\omega}\rangle$ has no repetitions. A contradiction. $\square$

Thus, we have proved the following, which answers a question form [9].

Theorem 7. *Suppose V is a model of ZFC plus MA_{ω_1}(σ-linked) and r is a random real over V. Then, $V[r] \models \neg MA_{\omega_1}$(Precalibre-$\aleph_1$).*

And the following answers a question from [1].

Corollary 8. *ZFC plus $MA_{\aleph_1}$(σ-linked) plus "Every Aronszajn tree is special", plus "Every (ω_1, ω_1)-gap on $\mathcal{P}(\omega)/Fin$ is indestructible" does not imply $MA_{\aleph_1}$(Precalibre-$\aleph_1$).*

Proof. This follows from the Theorem above and the fact that the theory ZFC plus $MA_{\aleph_1}$(σ-linked) plus "Every Aronszajn tree is special", plus "Every (ω_1, ω_1)-gap on $\mathcal{P}(\omega)/Fin$ is indestructible" is preserved by adding a random real, by results of Roitman [9], Laver [8], and Hirschorn [5]. $\square$

4. Some remarks on Cohen forcing

Roitman [9] (see also [10]) proves that $MA_{\aleph_1}$(σ-centered) is preserved by adding a Cohen real. But, as shown by Shelah [11], adding a Cohen real does not preserve $MA_{\aleph_1}$(σ-linked). Moreover, unlike the random real case, adding a Cohen real adds a Suslin tree (Shelah [11]) and an indestructible (ω_1, ω_1)-gap on $\mathcal{P}(\omega)/Fin$ (Todorčević). The arguments of the last section can be adapted to show that adding a Cohen real does not preserve $MA_{\aleph_1}$(Precalibre-$\aleph_1$) either. Indeed, let $\mathbb{C}$ be the Cohen poset, and let $\underset{\sim}{c}$ be the canonical name for the Cohen generic real added by $\mathbb{C}$. Letting $\underset{\sim}{\mathbb{P}} = \underset{\sim}{\mathbb{P}}_{\mathbb{C}, \underset{\sim}{c}}$, and $\underset{\sim}{I}_\delta$ be as before, one can show the following.

Lemma 9. *Suppose $MA_{\aleph_1}$(σ-centered) holds. Then in $V^{\mathbb{C}}$ there is no directed $\underset{\sim}{G} \subseteq \underset{\sim}{\mathbb{P}}$ such that for all $\delta \in \Omega$, $\underset{\sim}{I}_\delta \cap \underset{\sim}{G} \neq \emptyset$. Moreover, there are no $\underset{\sim}{m}$ and stationary $\underset{\sim}{S} \subseteq \omega_1$ such that $\underset{\sim}{S} \times \{\underset{\sim}{m}\} \subseteq \bigcup \underset{\sim}{G}$.*

The argument is entirely analogous to the proof of Lemma 5, the only difference being the use in Claim 6 of the *Amoeba for category* partial ordering, instead of $Amoeba_{\frac{1}{n}}$, and the fact that it is σ-centered. The Lemma yields the following theorem, a result of Kunen mentioned in [9] without proof.

Theorem 10. *Suppose V is a model of ZFC plus MA_{ω_1}(σ-centered) and c is a Cohen real over V. Then, $V[c] \models \neg MA_{\omega_1}$(Precalibre-$\aleph_1$).*

5. On destroying precalibre-$\aleph_1$ while preserving the ccc

In [13], J. Steprans and S. Watson ask the following.

Question 1. *Is it consistent that there is a precalibre-$\aleph_1$ poset which is ccc but does not have precalibre-$\aleph_1$ in some forcing extension that preserves cardinals?*

Note that the forcing extension cannot be ccc, since ccc forcing preserves the precalibre-$\aleph_1$ property. Moreover, as shown in [13], assuming MA_{ω_1} plus the Covering Lemma, every forcing that preserves cardinals also preserves the precalibre-$\aleph_1$ property.

A positive answer to Question 1 is provided by the following theorem. But first, let us recall the following strong form of Jensen's diamond principle, known as *diamond-star* relativized to a stationary set S, which is also due to Jensen. For S a stationary subset of ω_1, let

$\diamondsuit^*_S$: There exists a sequence $\langle \mathcal{S}_\alpha : \alpha \in S\rangle$, where $\mathcal{S}_\alpha$ is a countable set of subsets of α, such that for every $X \subseteq \omega_1$ there is a club $C \subseteq \omega_1$ with $X \cap \alpha \in \mathcal{S}_\alpha$, for every $\alpha \in C \cap S$.

The principle $\diamondsuit^*_S$ holds in the constructible universe L, for every stationary $S \subseteq \omega_1$ (see [3], 3.5, for a proof in the case $S = \omega_1$, which can be easily adapted to any stationary S). Also, $\diamondsuit^*_S$ can be forced by a σ-closed forcing notion (see [7], Chapter VII, Exercises H18 and H20, where it is shown how to force the even stronger form of diamond known as $\diamondsuit^+_S$).

Theorem 11. *It is consistent, modulo ZFC, that the CH holds and there exist*

(1) *A forcing notion T of cardinality $\aleph_1$ that preserves cardinals.*
(2) *Two posets $\mathbb{P}_0$ and $\mathbb{P}_1$ of cardinality $\aleph_1$ that have precalibre-$\aleph_1$ and such that*

$$\Vdash_T \text{“}\mathbb{P}_0, \mathbb{P}_1 \textit{ are ccc, but } \mathbb{P}_0 \times \mathbb{P}_1 \textit{ is not ccc.”}$$

Hence $\Vdash_T$ “$\mathbb{P}_0$ and $\mathbb{P}_1$ don't have precalibre-$\aleph_1$”.

Proof. Let $\{S_1, S_2\}$ be a partition of $\Omega := \{\delta < \omega_1 : \delta \text{ a limit}\}$ into two stationary sets. By a preliminary forcing, we may assume that $\diamondsuit^*_{S_1}$ holds. So, there exists $\langle \mathcal{S}_\alpha : \alpha \in S_1\rangle$, where $\mathcal{S}_\alpha$ is a countable set of subsets of α, such that for every $X \subseteq \omega_1$ there is a club $C \subseteq \omega_1$ with $X \cap \alpha \in \mathcal{S}_\alpha$, for every $\alpha \in C \cap S_1$. In particular, the CH holds. Using $\diamondsuit^*_{S_1}$, we can build an S_1-*oracle*, i.e., an $\subset$-increasing sequence $\bar{M} = \langle M_\delta : \delta \in S_1\rangle$, with M_δ countable and transitive, $\delta \in M_\delta$, $M_\delta \models$ “$ZFC^- + \delta$ is countable”, and such that for every $A \subseteq \omega_1$ there is a club $C_A \subseteq \omega_1$ such that $A \cap \delta \in M_\delta$, for every $\delta \in C_A \cap S_1$. (For the latter, one simply needs to require that $\mathcal{S}_\delta \subseteq M_\delta$, for all $\delta \in S_1$.) Moreover, we can build $\bar{M}$ so that it has the following additional property:

($*$) For every regular uncountable cardinal χ and a well ordering $<^*_\chi$ of $H(\chi)$, the set of all (universes of) countable $N \preceq \langle H(\chi), \in, <^*_\chi\rangle$ such that the Mostowski collapse of N belongs to M_δ, where $\delta := N \cap \omega_1$, is stationary in $[H(\chi)]^{\aleph_0}$.

To ensure this, take a big-enough regular cardinal λ and define the sequence $\bar{M}$ so that, for every $\delta \in S_1$, M_δ is the Mostowski collapse of a countable elementary substructure X of $H(\lambda)$ that contains $\bar{M} \restriction \delta$, all ordinals $\leq \delta$, and all elements of $\mathcal{S}_\delta$. To see that ($*$) holds, fix a regular uncountable cardinal χ, a well ordering $<^*_\chi$ of $H(\chi)$, and a club $E \subseteq [H(\chi)]^{\aleph_0}$. Let $\bar{N} = \langle N_\alpha : \alpha < \aleph_1\rangle$ be an $\subset$-increasing and $\in$-increasing continuous chain of elementary substructures of $\langle H(\chi), \in, <^*_\chi\rangle$ with the universe of N_α in E, for all $\alpha < \aleph_1$. We shall find $\delta \in S_1$ such that the transitive collapse of N_δ belongs to M_δ, where $\delta = N \cap \omega_1$.

Fix a bijection $h : \aleph_1 \to \bigcup_{\alpha<\aleph_1} N_\alpha$, and let $\Gamma : \aleph_1 \times \aleph_1 \to \aleph_1$ be the standard pairing function (cf. [6], 3). Observe that the set

$$D := \{\delta < \aleph_1 : \delta \text{ is closed under } \Gamma \text{ and } h \text{ maps } \delta \text{ onto } N_\delta\}$$

is a club. Now let

$$X_1 := \{\Gamma(i, j) : h(i) \in h(j)\}$$
$$X_2 := \{\Gamma(\alpha, i) : h(i) \in N_\alpha\}$$

$X_3 := \{\Gamma(i,j) : h(i) <^*_\chi h(j)\}$
$X := \{3j + i : i \in \{1,2,3\}\}$

The set $S_1' := \{\delta \in S_1 : X \cap \delta \in M_\delta\}$ is stationary. Thus, since the set $C := \{\delta < \aleph_1 : \delta = N_\delta \cap \omega_1\}$ is a club, we can pick $\delta \in C \cap D \cap S_1'$. Since $\delta \in D$, the structure

$$Y := \langle X_2 \cap \delta, \{\langle i,j\rangle : \Gamma(i,j) \in X_1 \cap \delta\}, \{\langle i,j\rangle : \Gamma(i,j) \in X_3 \cap \delta\}\rangle$$

is isomorphic to N_δ, and therefore Y and N_δ have the same transitive collapse. And since $\delta \in S_1'$, Y belongs to M_δ. Hence, since $M_\delta \models ZFC^-$, the transitive collapse of Y belongs to M_δ. Finally, since $\delta \in C$, $\delta = N_\delta \cap \omega_1$.

We shall define now the forcing T. Let us write $\aleph_1^{<\aleph_1}$ for the set of all countable sequences of countable ordinals. Let

$T := \{\eta \in \aleph_1^{<\aleph_1} : Range(\eta) \subset S_1$, η is increasing and continuous, of successor length, and if $\varepsilon < lh(\eta)$, then $\eta \restriction \varepsilon \in M_{\eta(\varepsilon)}\}$.

Let $\leq_T$ be the partial order on T given by end-extension. Thus, $(T, \leq_T)$ is a tree. Note that, since $\delta \in M_\delta$ for every $\delta \in S_1$, if $\eta \in T$, then $\eta \in M_{supRange(\eta)}$. Also notice that if $\eta \in T$, then $\eta^\frown\langle\delta\rangle \in T$, for every $\delta \in S_1$ greater than $supRange(\eta)$. In particular, every node of T of finite length has $\aleph_1$-many extensions of any bigger finite length. Now suppose $\alpha < \omega_1$ is a limit, and suppose, inductively, that for every successor $\beta < \alpha$, every node of T of length β has $\aleph_1$-many extensions of every higher successor length below α. We claim that every $\eta \in T$ of length less than α has $\aleph_1$-many extensions in T of length $\alpha+1$. For every $\delta < \omega_1$, let $T_\delta := \{\eta \in T : supRange(\eta) < \delta\}$. Notice that T_δ is countable: otherwise, uncountably-many $\eta \in T_\delta$ would have the same $supRange(\eta)$, and therefore they would all belong to the model $M_{supRange(\eta)}$, which is impossible because it is countable. Now fix a node $\eta \in T$ of length less than α, and let $B := \{b_\gamma : \gamma < \omega_1\}$ be an enumeration of all the *branches* (i.e., linearly-ordered subsets of T closed under predecessors) b of T that contain η and have length α (i.e., $\bigcup\{dom(\eta') : \eta' \in b\} = \alpha$). We shall build a sequence $B^* := \langle b^*_\xi : \xi < \omega_1\rangle$ of branches from B so that the set $supB^* := \langle supRange(\bigcup b^*_\xi) : \xi < \omega_1\rangle$ is the increasing enumeration of a club. To this end, start by fixing an increasing sequence $\langle \alpha_n : n < \omega\rangle$ of successor ordinals converging to α, with α_0 greater than the length of η. Then let $b^*_0 := b_0$. Given b^*_ξ, let γ be the least ordinal such that $\bigcup b_\gamma(\alpha_0) > supRange(\bigcup b^*_\xi)$, and let $b^*_{\xi+1} := b_\gamma$. Finally, given b^*_ξ for all $\xi < \delta$, where $\delta < \omega_1$ is a limit ordinal, pick an increasing sequence $\langle \xi_n : n < \omega\rangle$ converging to δ. If $\delta \in S_1$, then since $M_\delta \models$ "δ is countable", we pick $\langle \xi_n : n < \omega\rangle$ in M_δ. By construction, the sequence $\langle supRange(\bigcup b^*_{\xi_n}) : n < \omega\rangle$ is increasing. Now let $f : \alpha \to \aleph_1$ be such that $f \restriction [0, \alpha_0] = \bigcup b^*_{\xi_0} \restriction [0, \alpha_0]$, and $f \restriction (\alpha_n, \alpha_{n+1}] = \bigcup b^*_{\xi_{n+1}} \restriction (\alpha_n, \alpha_{n+1}]$, for all $n < \omega$. Then set $b^*_\zeta := \{f \restriction \beta : \beta < \alpha$ is a successor$\}$. One can easily check that b^*_ζ is a branch of T of length α with $supRange(\bigcup b^*_\zeta) = sup\{supRange(\bigcup b^*_\xi) : \xi < \zeta\}$.

By $(*)$ the set of all countable $N \preceq \langle H(\aleph_2), \in, <^*_{\aleph_2}\rangle$ that contain B^* and $\langle \alpha_n : n < \omega\rangle$, with $\alpha \subseteq N$, and such that the Mostowski collapse of N belongs to M_δ, where $\delta := N \cap \omega_1$, is stationary in $[H(\chi)]^{\aleph_0}$. So, since the set $Lim(supB^*)$ of limit points of $supB^*$ is a club, there is such an N with

$\delta := N \cap \omega_1 \in Lim(supB^*)$. If $\bar{N}$ is the transitive collapse of N, we have that $B^* \restriction \delta \in \bar{N} \in M_\delta$, and so in M_δ we can build, as above, the branch b^*_δ. Therefore, since $\delta = supRange(\bigcup b^*_\delta)$, we have that $\bigcup b^*_\delta \cup \{\langle \alpha, \delta \rangle\} \in T$ and extends η. We have thus shown that η has $\aleph_1$-many extensions in T of length $\alpha + 1$. Even more, the set $\{supRange(\bigcup b) : b$ is a branch of length $\alpha + 1$ that extends $\eta\}$ is stationary.

Note however that since the complement of S_1 is stationary, T has no branch of length ω_1, because the range of such a branch would be a club contained in S_1. But since every $\eta \in T$ has extensions of length $\alpha + 1$, for every α greater than or equal to the length of η, forcing with $(T, \geq_T)$ yields a branch of T of length ω_1.

In order to obtain the forcing notions $\mathbb{P}_0$ and $\mathbb{P}_1$ claimed by the theorem, we need first to force with the forcing $\mathbb{Q}$, which we define as follows. For u a subset of T, let $[u]^2_T$ be the set of all pairs $\{\eta, \nu\} \subseteq u$ such that $\eta \neq \nu$ and η and ν are $<_T$-comparable. Let

$$\mathbb{Q} := \{p : [u]^2_T \to \{0, 1\} : u \text{ is a finite subset of } T\},$$

ordered by reversed inclusion.

It is easily seen that $\mathbb{Q}$ is ccc, and it has cardinality $\aleph_1$, so forcing with $\mathbb{Q}$ does not collapse cardinals, does not change cofinalities, and preserves cardinal arithmetic. (In fact, $\mathbb{Q}$ is equivalent, as a forcing notion, to the poset for adding $\aleph_1$ Cohen reals, which is σ-centered, but we shall not make use of this fact.)

Notice that if $G \subseteq \mathbb{Q}$ is a generic filter over V, then $\bigcup G : [T]^2_T \to \{0, 1\}$.

Recall that, for $S \subseteq \aleph_1$ stationary, a forcing notion $\mathbb{P}$ is called *S-proper* if for all (some) large-enough regular cardinals χ and all (stationary-many) countable $\langle N, \in \rangle \preceq \langle H(\chi), \in \rangle$ that contain $\mathbb{P}$ and such that $N \cap \aleph_1 \in S$, and all $p \in \mathbb{P} \cap N$, there is a condition $q \leq p$ that is $(N, \mathbb{P})$-generic. If $\mathbb{P}$ is S-proper, then it does not collapse $\aleph_1$. (See [12], or [4] for details.)

Claim 12. *The forcing $\mathbb{Q} \times T$ is S_1-proper, hence it does not collapse $\aleph_1$.*

Proof of the claim. Let χ be a large-enough regular cardinal, and let $<^*_\chi$ be a well-ordering of $H(\chi)$. Let $N \preceq \langle H(\chi), \in, <^*_\chi \rangle$ be countable and such that $\mathbb{Q} \times T$ belongs to N, $\delta := N \cap \aleph_1 \in S_1$, and the Mostowski collapse of N belongs to M_δ. Fix $(q_0, \eta_0) \in (\mathbb{Q} \times T) \cap N$. It will be sufficient to find a condition $\eta_* \in T$ such that $\eta_0 \leq_T \eta_*$ and (q_0, η_*) is $(N, \mathbb{Q} \times T)$-generic.

Let

$$\mathbb{Q}_\delta := \{p \in \mathbb{Q} : \text{ if } \{\eta, \nu\} \in dom(p), \text{ then } \eta, \nu \in T_\delta\}.$$

Thus, $\mathbb{Q}_\delta$ is countable. Moreover, notice that $T_\delta = T \cap N$, and therefore $\mathbb{Q}_\delta = \mathbb{Q} \cap N$. Hence, T_δ and $\mathbb{Q}_\delta$ are the Mostowski collapses of T and $\mathbb{Q}$, respectively, and so they belong to M_δ.

In M_δ, let $\langle (p_n, D_n) : n < \omega \rangle$ list all pairs (p, D) such that $p \in \mathbb{Q}_\delta$, and D is a dense open subset of $\mathbb{Q}_\delta \times T_\delta$ that belongs to the Mostowski collapse of N. That is, D is the Mostowski collapse of a dense open subset of $\mathbb{Q} \times T$ that belongs to N.

Also in M_δ, fix an increasing sequence $\langle \delta_n : n < \omega \rangle$ converging to δ, and let

$$D'_n := \{(p, \nu) \in D_n : lh(\nu) > \delta_n\}.$$

Clearly, D'_n is dense open.

Note that, as the Mostowski collapse of N belongs to M_δ, we have that $(<^*_\chi \restriction (\mathbb{Q}_\delta \times T_\delta) = (<^*_\chi \restriction (\mathbb{Q} \times T)) \cap N \in M_\delta$.

Now, still in M_δ, and starting with (q_0, η_0), we inductively choose a sequence $\langle (q_n, \eta_n) : n < \omega \rangle$, with $q_n \in \mathbb{Q}_\delta$ and $\eta_n \in T_\delta$, and such that if $n = m+1$, then:

(a) $p_n \geq q_n$ and $\eta_m <_T \eta_n$.
(b) $(q_n, \eta_n) \in D'_n$.
(c) (q_n, η_n) is the $<^*_\chi$-least such that (a) and (b) hold.

Then, $\eta_* := (\bigcup_n \eta_n) \cup \{\langle \delta, \delta \rangle\} \in T$, and $\eta^* \in M_\delta$, hence $(q_0, \eta_*) \in \mathbb{Q} \times T$. Clearly, $(q_0, \eta_*) \leq (q_0, \eta_0)$. So, we only need to check that (q_0, η_*) is $(N, \mathbb{Q} \times T)$-generic.

Fix an open dense $E \subseteq \mathbb{Q} \times T$ that belongs to N. We need to see that $E \cap N$ is predense below (q_0, η_*). So, fix $(r, \nu) \leq (q_0, \eta_*)$. Since $\mathbb{Q}$ is ccc, q_0 is $(N, \mathbb{Q})$-generic, so we can find $r' \in \{p : (p, \eta) \in E, \text{ some } \eta\} \cap N$ that is compatible with r. Let n be such that $p_n = r'$ and D_n is the Mostowski collapse of E. Then (p_n, η_n) belongs to the transitive collapse of E, hence to $E \cap N$, and is compatible with (r, ν), as $(p_n, \eta_*) \leq (p_n, \eta_n)$. □

We thus conclude that if $G \subseteq \mathbb{Q}$ is a filter generic over V, then in $V[G]$ the forcing T does not collapse $\aleph_1$, and therefore, being of cardinality $\aleph_1$, it preserves cardinals, cofinalities, and the cardinal arithmetic.

We shall now define the $\mathbb{Q}$-names for the forcing notions $\underset{\sim}{\mathbb{P}}_\ell$, for $\ell \in \{0, 1\}$, as follows: in $V^{\mathbb{Q}}$, let $\underset{\sim}{b} = \bigcup \underset{\sim}{G}$, where $\underset{\sim}{G}$ is the standard $\mathbb{Q}$-name for the $\mathbb{Q}$-generic filter over V. Then let

$$\begin{aligned} \underset{\sim}{\mathbb{P}}_\ell := \{(w, c) :\ & w \subseteq T \text{ is finite, } c \text{ is a function from } w \text{ into } \omega \text{ such that if} \\ & \{\eta, \nu\} \in [w]^2_T \text{ and } \underset{\sim}{b}(\{\eta, \nu\}) = \ell, \text{ then } c(\eta) \neq c(\nu)\}. \end{aligned}$$

A condition (w, c) is stronger than a condition (v, d) if and only if $w \supseteq v$ and $c \supseteq d$.

We shall show that if G is $\mathbb{Q}$-generic over V, then in the extension $V[G]$, the partial orderings $\mathbb{P}_\ell = \underset{\sim}{\mathbb{P}}_\ell[G]$, for $\ell \in \{0, 1\}$, and T are as required.

Claim 13. *In $V[G]$, $\mathbb{P}_\ell$ has precalibre-$\aleph_1$.*

Proof of the claim. Assume $p_\alpha = (w_\alpha, c_\alpha) \in \mathbb{P}_\ell$, for $\alpha < \omega_1$. We shall find an uncountable $S \subseteq \aleph_1$ such that $\{p_\alpha : \alpha \in S\}$ is finite-wise compatible. For each $\delta \in S_2$, let

$$s_\delta := \{\eta \restriction (\gamma{+}1) : \eta \in w_\delta, \text{ and } \gamma \text{ is maximal such that } \gamma < lh(\eta) \wedge \eta(\gamma) < \delta\}.$$

As η is an increasing and continuous sequence of ordinals from S_1, hence disjoint from S_2, the set s_δ is well-defined. Notice that s_δ is a finite subset of $T_\delta := \{\eta \in T : supRange(\eta) < \delta\}$, which is countable.

Let $s^1_\delta := w_\delta \cap T_\delta$. Note that $s^1_\delta \subseteq s_\delta$.

Let $f : S_2 \to \omega_1$ be given by $f(\delta) = max\{supRange(\eta) : \eta \in s_\delta\}$. Thus, f is regressive, hence constant on a stationary $S_3 \subseteq S_2$. Let δ_0 be the constant value of f on S_3. Then, $s_\delta \subseteq T_{\delta_0}$, for every $\delta \in S_3$. So, since T_{δ_0} is countable,

there exist $S_4 \subseteq S_3$ stationary and s_* such that $s_\delta = s_*$, for every $\delta \in S_4$. Further, there is a stationary $S_5 \subseteq S_4$ and s^1_* and c_* such that for all $\delta \in S_5$,

$$s^1_\delta = s^1_*, \; c_\delta \restriction s^1_* = c_*, \text{ and } \forall \alpha < \delta (w_\alpha \subseteq T_\delta).$$

Hence, if $\delta_1 < \delta_2$ are from S_5, then not only $w_{\delta_1} \cap w_{\delta_2} = s^1_*$, but also if $\eta_1 \in w_{\delta_1} - s^1_*$ and $\eta_2 \in w_{\delta_2} - s^1_*$, then η_1 and η_2 are $<_T$-incomparable: for suppose otherwise, say $\eta_1 <_T \eta_2$. If $\gamma + 1 = lh(\eta_1)$, then $\eta_2 \restriction (\gamma+1) = \eta_1 <_T \eta_2$, and $\eta_2(\gamma) = \eta_1(\gamma) < \delta_2$, by choice of S_5. Hence, by the definition of s_{δ_2}, $\eta_2 \restriction (\gamma+1) = \eta_1$ is an initial segment of some member of $s_{\delta_2} = s_*$, and so it belongs to T_{δ_1}, hence $\eta_1 \in s^1_*$, contradicting the assumption that $\eta_1 \notin s^1_*$.

So, $\{p_\delta : \delta \in S_5\}$ is as required. □

It only remains to show that forcing with T over $V[G]$ preserves the ccc-ness of $\mathbb{P}_0$ and $\mathbb{P}_1$, but makes their product not ccc.

Claim 14. *If G_T is T-generic over $V[G]$, then in the generic extension $V[G][G_T]$, the forcing $\mathbb{P}_\ell$ is ccc.*

Proof of the claim. First notice that, by the Product Lemma (see [6], 15.9), G is $\mathbb{Q}$-generic over $V[G_T]$, and $V[G][G_T] = V[G_T][G]$. Now suppose $\underset{\sim}{A} = \{(\underset{\sim}{w}_\alpha, \underset{\sim}{c}_\alpha) : \alpha < \omega_1\} \in V[G_T]$ is a $\mathbb{Q}$-name for an uncountable subset of $\mathbb{P}_\ell$. For each $\alpha < \omega_1$, let $p_\alpha \in \mathbb{Q}$ and (w_α, c_α) be such that $p_\alpha \Vdash$ "$(\underset{\sim}{w}_\alpha, \underset{\sim}{c}_\alpha) = (w_\alpha, c_\alpha)$". Let u_α be such that $dom(p_\alpha) = [u_\alpha]^2_T$. By extending p_α, if necessary, we may assume that $w_\alpha \subseteq u_\alpha$, for all $\alpha < \omega_1$. We shall find $\alpha \neq \beta$ and a condition p that extends both p_α and p_β and forces that (w_α, c_α) and (w_β, c_β) are compatible. For this, first extend (w_α, c_α) to (u_α, d_α) by letting d_α give different values in $\omega \setminus Range(c_\alpha)$ to all $\eta \in u_\alpha \setminus w_\alpha$. We may assume that the set $\{u_\alpha : \alpha < \omega_1\}$ forms a Δ-system with root r. Moreover, we may assume that p_α restricted to $[r]^2_T$ is the same for all $\alpha < \omega_1$, and also that d_α restricted to r is the same for all $\alpha < \omega_1$. Now pick $\alpha \neq \beta$ and let $p : [u_\alpha \cup u_\beta]^2_T \to \{0,1\}$ be such that $p \restriction [u_\alpha]^2_T = p_\alpha$, $p \restriction [u_\beta]^2_T = p_\beta$, and $p(\{\eta, \nu\}) \neq \ell$, for all other pairs in $[u_\alpha \cup u_\beta]^2_T$. Then, p extends both p_α and p_β, and forces that (u_α, d_α) and (u_β, d_β) are compatible, hence it forces that (w_α, c_α) and (w_β, c_β) are compatible. □

But in $V[G][G_T]$, the product $\mathbb{P}_0 \times \mathbb{P}_1$ is not ccc. For let $\eta^* = \bigcup G_T$. For every $\alpha < \omega_1$, let $p^\ell_\alpha := (\{\eta^* \restriction (\alpha+1)\}, c^\ell_\alpha) \in \mathbb{P}_\ell$, where $c^\ell_\alpha(\eta^* \restriction (\alpha+1)) = 0$. Then the set $\{(p^0_\alpha, p^1_\alpha) : \alpha < \omega_1\}$ is an uncountable antichain. □

We finish with two well-known open questions

Question 2. *Does $MA_{\aleph_1}$(Powerfully-ccc) imply $MA_{\aleph_1}$?*

Question 3. *Does "Every ccc poset has precalibre-$\aleph_1$" imply $MA_{\aleph_1}$?*

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